UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH1201**

ASSESSMENT PATTERN : **MATH1201A**

MODULE NAME : **Algebra 1**

DATE : **13-May-10**

TIME : **10:00**

TIME ALLOWED : **2 Hours 0 Minutes**

2009/10-MATH1201A-001-EXAM-198

TURN OVER

All questions may be attempted but only marks obtained on the best four *solutions will count.*
The use of an electronic calculator is not *permitted in this examination.*

1. (i) If $\mathcal{S} = (q \vee \neg p) \bigwedge (p \wedge \neg q)$ find a formula equivalent to $\neg\mathcal{S}$ which does not involve $\wedge, \vee$ or $\neg$.

   Without computing truth tables decide which of the following (a), (b) or (c) is correct:

   (a) $\mathcal{S}$ is a tautology ; (b) $\mathcal{S}$ is a contradiction ; (c) $\mathcal{S}$ is neither a tautology nor a contradiction.

   (ii) Find an equivalent formula to $\mathcal{T}$ below which does not involve $\exists$, $\neg$ or $\Longrightarrow$.
   $\mathcal{T} = \neg(\exists x)\,(P(x) \Longrightarrow ((\forall y)R(y) \Longrightarrow (\exists z)\neg S(z)))$.

   (iii) Let $f : A \to B$ be a mapping between sets A, B. Explain what is meant by saying that

   (a) f is injective ; (b) f is surjective ; (c) f is invertible.

   Prove that f is invertible if and only if f is both injective and surjective.

2. Let $\epsilon(r,s)$ be the $n \times n$ matrix given by $\epsilon(r,s)_{ij} = \delta_{ri}\delta_{sj}$ where 'δ' denotes the Kronecker delta. Prove that

$$\epsilon(r,s)\epsilon(p,t) = \begin{cases} \epsilon(r,t) & \text{if } s = p \\ 0 & \text{if } s \neq p \end{cases}$$

   The elementary matrices $\Delta(r,\alpha)$ and $E(r,s;\lambda)$ $(r \neq s)$ are defined by

$$\Delta(r,\alpha) = I_n + (\alpha - 1)\epsilon(r,r) \;\; ; \;\; E(r,s;\lambda) = I_n + \lambda\epsilon(r,s).$$

   When $r \neq s$ express (in terms of I_n, $\epsilon(r,r)$ and $\epsilon(r,s)$) the matrix products

   (i) $E(r,s;\mu)\Delta(r,\alpha)$ and (ii) $\Delta(r,\alpha)E(r,s;\lambda)$.

   If $\Delta(r,\alpha)E(r,s;\lambda) = E(r,s;\mu)\Delta(r,\alpha)$ express μ in terms of α, λ.

   For the matrix A below, find A^{-1}. Moreover, by first expressing A^{-1} as a product of elementary matrices also express A as a product of elementary matrices.

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

3. Let V be a vector space over a field $\mathbb{F}$ and let $\{\mathrm{v}_1, \ldots, \mathrm{v}_n\}$ be a subset of V ; explain what is meant by saying that $\{\mathrm{v}_1, \ldots, \mathrm{v}_n\}$ is linearly independent over $\mathbb{F}$.

In each case below, decide with justification whether the given vectors are linearly independent over $\mathbb{Q}$. If they are not, give an explicit dependence relation between them.

(a) $$\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \\ -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix};$$

(b) $$\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}.$$

Explain what is meant by a *spanning set* for a vector space V. Let $\{\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n\}$ be a spanning set for V, and suppose that $\mathrm{u} \in V$ can be expressed as a linear combination of the form

$$\mathbf{u} = \sum_{r=1}^{n} \lambda_r \mathrm{v}_r \quad .$$

with $\lambda_1 \neq 0$. Show that $\{\mathrm{u}, \mathrm{v}_2, \ldots, \mathrm{v}_n\}$ is also a spanning set for V.

State and prove the Exchange Lemma.

4. Let V, W be vector spaces over a field $\mathbb{F}$ and let $T : V \to W$ be a mapping; explain what is meant by saying that T is *linear*.

When T is linear, explain what is meant by

(a) the kernel, $\mathrm{Ker}(T)$ and

(b) the image, $\mathrm{Im}(T)$.

State and prove a relationship which holds between $\dim \mathrm{Ker}(T)$ and $\dim \mathrm{Im}(T)$.

Let $T_A : \mathbb{Q}^6 \to \mathbb{Q}^4$ be the linear mapping $T_A(\mathrm{x}) = A\mathrm{x}$, where

$$A = \begin{pmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & -1 & 2 & 2 \\ 3 & -4 & 1 & 2 & 1 & 1 \end{pmatrix}.$$

Find (i) $\dim \mathrm{Ker}(T_A)$; (ii) a basis for $\mathrm{Ker}(T_A)$; (iii) a basis for $\mathrm{Im}(T_A)$.

5. Let V be the vector space consisting of all functions $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = e^{-x}\{a_0 + a_1 x + a_2 x^2\} \quad (a_i \in \mathbb{Q})$$

and let $D : V \to V$ be the linear map $D(f) = \frac{df}{dx}$.

Taking $\{e^{-x}, xe^{-x}, x^2e^{-x}\}$ as basis for V find :

(i) the matrix of D ; (ii) the matrix of D^4 ; (iii) the matrix of D^{-1}.

Hence *without further explicit differentiation or integration* write down

(iv) $\frac{d^4}{dx^4}(e^{-x} + \frac{x^2e^{-x}}{4})$ (v) $\int \{e^{-x} - 5xe^{-x} + 2x^2e^{-x}\}\, dx$

[You may ignore the constant of integration in (v)].

6. Let $T : U \to V$ be a linear map between vector spaces U, V, and let $\mathcal{E} = (e_i)_{1\leqslant i\leqslant m}$ be a basis for U and $\Phi = (\varphi_j)_{1\leqslant j\leqslant n}$ be a basis for V.

Explain what is meant by the matrix $\mathcal{M}(T)^{\Phi}_{\mathcal{E}}$ of T taken with respect to $\mathcal{E}$ (on the left) and Φ (on the right).

Now suppose that $V = U$:

(i) prove that $\mathcal{M}(\mathrm{Id}_U)^{\Phi}_{\mathcal{E}}$ is invertible and give an expression for $(\mathcal{M}(\mathrm{Id}_U)^{\Phi}_{\mathcal{E}})^{-1}$;

(ii) state and prove a relationship which holds between $\mathcal{M}(T)^{\Phi}_{\Phi}$ and $\mathcal{M}(T)^{\mathcal{E}}_{\mathcal{E}}$.

Consider the following two bases for $\mathbb{Q}^3$;

$$\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \; ; \; \Phi = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \right\}$$

and let $T : \mathbb{Q}^3 \to \mathbb{Q}^3$ be the mapping $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 & +x_2 & +8x_3 \\ x_1 & +x_2 & -3x_3 \\ & x_2 & +3x_3 \end{pmatrix}$.

Write down $\mathcal{M}(T)^{\mathcal{E}}_{\mathcal{E}}$ and find $\mathcal{M}(T)^{\Phi}_{\Phi}$.